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On Global Geometric Properties of Subspaces in Hilbert Space*

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Some global properties of the set of all direct complements to a given subspace in a Hilbert space are studied.

1. INTRODUCTION

Let H be a separable Hilbert space. For any pair of (closed) subspaces A, $B \subset H$, define

$$\delta(A, B) = \max\{\sup_{x \in S_A} \operatorname{dist}(x, B), \sup_{y \in S_B} \operatorname{dist}(y, B)\},$$

where $S_A(S_B)$ is the unit sphere in $A(B)$, and $\operatorname{dist}(z, M) = \min_{m \in M} \|z - m\|$ is the distance from $z \in H$ to the subspace M. (We put $\delta(A, 0) = \delta(0, A) = 1$ for $A \neq 0$ and $\delta(0, 0) = 1$.) The function $\delta(A, B)$ is called *gap* (or aperture) between the subspaces A and B; $\delta(A, B)$ is a metric, and in the topology induced by this metric, the set $\Omega = \Omega(H)$ of all subspaces in H is a complete metric space ([21, see also [3, p. 198]).

The metric space Ω has been studied by several authors ([1–3, 5] is a sample of related works). However, in many cases the attention was confined to he local properties of Ω. The following well-known fact is an example of such a local approach (see, for instance, [2, Theorem 2]).

For any pair of complementary subspaces A and B in H ($A \dot{+} B = H$) there exists $\varepsilon > 0$ such that $A \dot{+} \tilde{B} = H$ holds for any subspace $\tilde{B} \subset H$ with the property that $\delta(B, \tilde{B}) < \varepsilon$. (Here and elsewhere the notation $A \dot{+} B$ means that the sum $A + B$ is direct, i.e., $A \cap B = \{0\}$ and $A + B$ is closed.)

The purpose of this paper is to study some global properties of the metric

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space Ω. In particular, Theorem 1 below can be viewed as a global version of the result stated in the preceding paragraph.

In the next section we present the main results of the paper.

2. Main Results

Let us introduce first the necessary definitions. For given m, n (which can be nonnegative integers or infinity) such that $m+n=\dim H$, let

$$\Omega(m,n)=\{A\in\Omega(H)|\dim A=m;\ \operatorname{codim} A=n\}.$$

As is well known (for example, [3, Corollary IV.2.6]), the set $\Omega(m,n)$ is open and closed in Ω; so that Ω is a disconnected union of all the complete metric spaces $\Omega(m,n)$, where $m+n=\dim H$.

For every subspace $A\subset H$, we denote by $D_A\subset\Omega$ the set of all direct complements B to A in H: $A\dot{+}B=H$. As it follows from the result stated in the introduction, the set D_A is open in Ω. For $A\in\Omega(m,n)$ denote by E_A the set of all subspaces $B\in\Omega(n,m)$ such that $\dim A\cap B=\operatorname{codim}\overline{A+B}$. Clearly, $D_A\subset E_A$. If $m<\infty$ or $n<\infty$ then $E_A=\Omega(n,m)$.

Theorem 1. *Let $A\in\Omega(m,n)$. Then the set D_A is dense and connected in E_A.*

As we shall see later (Example 1 in Section 4), in Theorem 1 one cannot replace E_A by $\Omega(n,m)$ in general.

In the finite dimensional case ($\dim H=p<\infty$) Theorem 1 is easily obtained. Indeed, let $x_1,\dots,x_m\in\mathbb{C}^p$ be a basis in A; then the subspace $\operatorname{span}\{y_1,\dots,y_n\}$ spanned by some vectors $y_1,\dots,y_n\in\mathbb{C}^p$ is in D_A if and only if

$$\det[x_1\cdots x_m\,y_1\cdots y_n]\neq 0. \tag{1}$$

Consider $\det[x_1\cdots x_m\,y_1\cdots y_n]$ as a polynomial in np independent complex variables (the coordinates of $y_1,\dots,y_n$); then (see, for example, [4, Theorem III.C.16] the set of all points in $\mathbb{C}^{np}$ (each point representing the coordinates of $y_1,\dots,y_n$) such that (1) holds, is dense and connected in $\mathbb{C}^{np}$. Denote by $\Psi_{n,p}$ the set of all linearly independent systems of n p-dimensional vectors (with topology induced from $\mathbb{C}^{np}$); the map φ: $\Psi_{n,p}\to\Omega(\mathbb{C}^p)$ defined by the equality $\varphi(z_1,\dots,z_n)=\operatorname{span}\{z_1,\dots,z_n\}\subset\mathbb{C}^p$ for every system $(z_1,\dots,z_n)\in\Psi_{n,p}$, is continuous (Ref. [5, Chap. IV.7], and taking into account the preceding observations, we finish the proof of Theorem 1 for $H=\mathbb{C}^p$.

In case $\dim H=\infty$, we shall use completely different considerations to prove Theorem 1. The proof of Theorem 1 will be given in Section 4.

For given $A \in \Omega$, consider the set of all $B \in \Omega$ such that the sum $A + B$ is closed. It turns out that this set is dense in Ω, as shows the following result.

THEOREM 2. *Let B and A be subspaces in H. Then for every $\varepsilon > 0$ there exists a subspace B_ε in H such that*

$$\lim_{\varepsilon \to 0} \delta(B, B_\varepsilon) = 0 \tag{2}$$

and the linear set $A + B_\varepsilon$ is closed for every $\varepsilon > 0$.

Proof. We shall reduce the proof to the usage of Theorem 1. If dim $A < \infty$ or codim $A < \infty$, then clearly $A + B_\varepsilon$ is closed for every subspace $B_\varepsilon \subset H$. So suppose $A \in \Omega(\infty, \infty)$. We can (and will) assume also that $B \in \Omega(\infty, \infty)$ (otherwise take $B_\varepsilon = B$). Consider several cases:

(1) $\dim A \cap B = \operatorname{codim} \overline{A + B} = 0$; so $B \in E_A$. Now B_ε can be found in view of Theorem 1 (in this case even $A + B_\varepsilon = H$).

(2) $A \cap B \neq \{0\}$; $\overline{A + B} = H$. Let $A' = A \ominus (A \cap B)$; $B' = B \ominus (A \cap B)$; $H' = H \ominus (A \cap B)$. In view of case (1) for every $\varepsilon > 0$ there exists a subspace $B'_\varepsilon \subset H'$ such that $\delta(B'_\varepsilon, B') < \varepsilon$ and $A' + B'_\varepsilon$ is closed in H'. Now put $B_\varepsilon = B'_\varepsilon \oplus B \cap A$; by Proposition 5 (ii) (see Section 3), (2) is satisfied.

(3) $A + B \neq H$. In this case replace H by $\overline{A + B}$ to reduce the proof to the case (1) or (2). ∎

Consider now the problem of existence of common direct complements. If $\min(m, n) < \infty$ it is easily solved as shows the following corollary.

COROLLARY 3. *Let $A_p \in \Omega(m, n)$, $p = 1, 2, \ldots$, where at least one of the numbers m, n is finite. Then there exists a common direct complement to A_p, i.e., a subspace $B \subset H$ such that*

$$B \dot{+} A_p = H, \qquad p = 1, 2, \ldots \tag{3}$$

Moreover, the set $\bigcap_{i=1}^{\infty} D_{A_i}$ of all $B \in \Omega(n, m)$ such that (3) *holds, is dense in $\Omega(n, m)$.*

Proof. Suppose the contrary; then $\bigcap_{i=1}^{\infty} D_{A_i} = \varnothing$. So $\bigcup_{i=1}^{\infty} (\Omega(n, m) \backslash D_{A_i}) = \Omega(n, m)$. Since $\Omega(n, m)$ is complete, by Baire's category theorem, at least one of the closed sets $\Omega(n, m) \backslash D_{A_i}$ contains an interior point; let it be $\Omega(n, m) \backslash D_{A_i}$. But then D_{A_i} is not dense in $\Omega(n, m)$, a contradiction with Theorem 1. The second part of Corollary 3 follows from the well-known fact that the intersection of countably many dense open subsets in a complete metric space is also dense. ∎

For the case $m = n = \infty$ we prove the following result.

THEOREM 4. *Let A, $B \in \Omega(\infty, \infty)$. Then for every $\varepsilon > 0$ there exist subspaces $B_\epsilon, C_\epsilon \in \Omega(\infty, \infty)$ such that $\delta(B, B_\epsilon) < \varepsilon$ and $A \dot{+} C_\epsilon = B_\epsilon \dot{+} C_\epsilon = H$.*

Note that not every pair of subspaces A, $B \in \Omega(\infty, \infty)$ has a common direct complement in H (for instance, this is the case when $A \subset B$ and $A \neq B$). Theorem 4 states, in particular, that the pairs of subspaces which admit a common direct complement, form a dense set.

3. AUXILIARY RESULTS

In this section we present some auxiliary results which will be needed in the proof of Theorem 1. As before, H stands for a separable Hilbert space.

We begin with more properties of the gap function. The notation $H_1 \oplus H_2$ is used for the orthogonal sum of separable Hilbert spaces H_1 and H_2.

PROPOSITION 5. (i) *For a given subspace $A \subset H$, the set D_A of all subspaces $X \subset H$ such that $X \dot{+} A = H$, is open in $\Omega(H)$;*

(ii) *Let $H = H_1 \oplus H_2$, and let $A \subset H_1$ be a subspace. Then for $\varepsilon > 0$ small enough*

$$\delta(A_\epsilon \dot{+} H_2, A \oplus H_2) < 2\varepsilon,$$

where $A_\epsilon \subset H$ is an arbitrary subspace such that $\delta(A, A_\epsilon) < \varepsilon$;

(iii) *For $A, B \in \Omega(H)$,*

$$\delta(A, B) = \delta(A^\perp, B^\perp).$$

Proof. (i) has been mentioned already; it follows readily from the result stated in the introduction.

Part (iii) is found in [3, Theorem IV.2.9].

We prove (ii). In view of (i), $A_\epsilon \dot{+} H_2$ is indeed a direct sum for $\varepsilon > 0$ small enough. Further,

$$S_{A \oplus H_2} = \{a + h_2 \in A \oplus H_2 \mid \|a\|^2 + \|h_2\|^2 = 1\};$$

and for $a + h_2 \in S_{A \oplus H_2}$ we have

$$\begin{aligned}\operatorname{dist}(a + h_2, A_\epsilon \dot{+} H_2) &\leqslant \min_{a_\epsilon \in A_\epsilon} \|a - a_\epsilon\| = \operatorname{dist}(a, A_\epsilon) \\ &\leqslant \|a\|\, \delta(A, A_\epsilon) < \varepsilon.\end{aligned}$$

$\|a_\varepsilon - a\| \leqslant \|a_\varepsilon\|\,\delta(A, A_\varepsilon) < \varepsilon\|a_\varepsilon\|$ for some $a \in A$. Therefore $\|a\| \geqslant (1-\varepsilon)\|a_\varepsilon\|$ and

$$1 = \|a_\varepsilon + h_2\| \geqslant \|a + h_2\| - \|a_\varepsilon - a\| \geqslant \|a\| - \varepsilon\|a_\varepsilon\| \geqslant (1-2\varepsilon)\|a_\varepsilon\|,$$

or $\|a_\varepsilon\| \leqslant (1-2\varepsilon)^{-1}$. Now

$$\operatorname{dist}(a_\varepsilon + h_2, A \oplus H_2) \leqslant \min_{a\in A} \|a_\varepsilon - a\| = \|a_\varepsilon\| \cdot \operatorname{dist}\left(\frac{a_\varepsilon}{\|a_\varepsilon\|}, A\right)$$
$$\leqslant \varepsilon(1-2\varepsilon)^{-1}.$$

By the definition of the gap function, (ii) follows for $\varepsilon > 0$ such that $(1-2\varepsilon)^{-1} < 2$, or $\varepsilon < \frac{1}{4}$. ∎

LEMMA 6. *For every pair of subspaces A, $B \subset H$ such that $\dim A = \dim B$ and $A^\perp \cap B = B^\perp \cap A = \{0\}$, there exists a selfadjoint operator $Z \in L(H)$ with the properties $Z(A) = B$ and* $\operatorname{Ker} Z|_A = \{0\}$.

Proof. Let $e_1, e_2, \ldots,$ (resp. $f_1, f_2, \ldots,$) be an orthonormal basis in A (resp. B). Define linear operators $X_0 : A \to A$, $Y_0 : A \to A^\perp$:

$$X_0 e_i = P_A f_i, \qquad Y_0 e_i = (I - P_A) f_i, \qquad i = 1, 2, \ldots, \tag{4}$$

where P_A is the orthogonal projector on A. Let $X_0 = QU$ be the polar representation of X_0, where $U \in L(A)$ is a partial isometry which maps $\overline{\operatorname{Im} X_0}$ isometrically onto $\overline{\operatorname{Im} X_0^*}$ and $Q \geqslant 0$. It is not hard to see that

$$\operatorname{Ker} X_0 = \{0\}, \qquad \overline{\operatorname{Im} X_0} = A. \tag{5}$$

Indeed, let $x = \sum_{i=1}^\infty a_i e_i \in \operatorname{Ker} X_0$. Then $P_A(\sum_{i=1}^\infty a_i f_i) = 0$, i.e., $\sum_{i=1}^\infty a_i f_i \in A^\perp \cap B = \{0\}$. So $a_i = 0$ and $x = 0$. If $\overline{\operatorname{Im} X_0} \neq A$, then there exists a non-zero vector $y \in A$ which is orthogonal to $\operatorname{Im} X_0$. So $0 = (y, P_A f_i) = (P_A y, f_i) = (y, f_i)$, i.e., $y \in B^\perp$. But $B^\perp \cap A = \{0\}$, so $y = 0$. We obtain that $U: A \to A$ is in fact unitary. Now put

$$Z = \begin{bmatrix} Q & (Y_0 U^{-1})^* \\ Y_0 U^{-1} & 0 \end{bmatrix} : A \oplus A^\perp \to A \oplus A^\perp.$$

Clearly, $Z = Z^*$, and since

$$Z|_A = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix} U^{-1},$$

the operator Z satisfies requirements of Lemma 6. ∎

4. Proof of Theorem 1

We prove first the connectedness of D_A, which is the easy part of Theorem 1 (the auxiliary results are not used here). The proof is based on the simple observation that $B \in D_A$ if and only if B is the image of an operator of the form $\left[\begin{smallmatrix} X \\ I \end{smallmatrix}\right]: A^\perp \to A \oplus A^\perp$, where I denotes the identity in $L(A^\perp)$. Indeed, $\mathrm{Im}\left[\begin{smallmatrix} X \\ I \end{smallmatrix}\right]$ is obviously a direct complement to A in H. Conversely, for $B \in D_A$ it is easy to check that the restriction $P|_B : B \to A$, where P is the orthogonal projector on $A^\perp$, is invertible. Then the equality $B = \mathrm{Im}\left[\begin{smallmatrix} X \\ I \end{smallmatrix}\right]$ holds for $X = (I-P)|_B \cdot (P|_B)^{-1}: A^\perp \to A$. The set $L(A^\perp, A)$ of all (linear bounded) operators $X: A^\perp \to A$ is clearly connected, so in order to prove the connectedness of D_A, we have only to show that the map $\varphi: L(A^\perp, A) \to D_A$ defined as $\varphi X = \mathrm{Im}\left[\begin{smallmatrix} X \\ I \end{smallmatrix}\right]$, is continuous. To this end recall the inequality (see [1]):

$$\delta(B_1, B_2) \leqslant \|P_1 - P_2\|, \tag{6}$$

where B_1, B_2 are subspaces in H, and P_i is a projector (not necessarily orthogonal) on B_i, $i = 1, 2$. Use this inequality for

$$B_i = \mathrm{Im} \begin{bmatrix} X_i \\ I \end{bmatrix}$$

and

$$P_i = \begin{bmatrix} 0 & X_i \\ 0 & I \end{bmatrix} \qquad (i = 1, 2)$$

to deduce the continuity of φ.

We pass now to the proof of denseness of D_A in E_A (under assumption that $\dim H = \infty$). Several cases can occur:

(a) $\dim A = m < \infty$; then $n = \infty$. Let $B \in \Omega(\infty, m)$ $(= E_A)$ be arbitrary. We have to prove that for every $\varepsilon > 0$ there exists $B_\varepsilon \in D_A$ such that $\lim_{\varepsilon \to 0} \delta(B, B_\varepsilon) = 0$. Let $p = \dim(A \cap B)$; then also $\dim C = p$, where $C = (A + B)^\perp$. Let $e_1, \ldots, e_p$ (resp. $f_1, \ldots, f_p$) be anorthonormal basis in $A \cap B$ (resp. C), and put

$$B_\varepsilon = (B \ominus (A \cap B) \dotplus \mathrm{span}\{e_i + \varepsilon f_i\}_{i=1}^p, \tag{7}$$

where $\varepsilon > 0$ is sufficiently close to zero.

Let us check that B_ε defined by (7) is the desired subspace, i.e., $B_\varepsilon \in D_A$ and $\lim_{\varepsilon \to 0} \delta(B, B_\varepsilon) = 0$. The latter equality follows from Proposition 5(ii). To prove that B_ε is a direct complement to A in H, it is sufficient to check that $A \cap B_\varepsilon = \{0\}$ (because $\dim A = m$ and $B_\varepsilon \in \Omega(\infty, m)$ for ε close enough to 0). Let $a = a_1 + a_2 \in A$, where $a_1 \in A \cap B$ and $a_2 \in A \ominus (A \cap B)$, and suppose that $a \in B_\epsilon$:

$$a = b + \sum_{i=1}^{p} \alpha_i(e_i + \varepsilon f_i), \tag{8}$$

where $\alpha_i \in \mathbb{C}$, $b \in B \ominus (A \cap B)$. So

$$a_1 - \sum_{i=1}^{p} \alpha_i e_i = b - a_2 + \varepsilon \sum_{i=1}^{p} \alpha_i f_i.$$

The left-hand side belongs to $A \cap B$, while the right-hand side is orthogonal to $A \cap B$. Therefore $a_1 = \sum_{i=1}^{p} \alpha_i e_i$, $a_2 - b = \varepsilon \sum_{i=1}^{p} \alpha_i f_i$. In the latter equality the left-hand side belongs to $A + B$ and the right-hand side belongs to $C = (A + B)^\perp$, so $a_2 = b$ and $\sum_{i=1}^{p} \alpha_i f_i = 0$. It follows that $\alpha_1 = \cdots = \alpha_p = 0$. Then (8) gives

$$a = b \in [A \ominus (A \cap B)] \cap [B \ominus (A \cap B)] = 0,$$

so $a = 0$ and $A \cap B_\epsilon = \{0\}$. Thus Theorem 1 is proved in the case that dim $A = m < \infty$.

(b) The case codim $A = n < \infty$ can be reduced to the preceding one, by considering $A^\perp$ in place of A and using Proposition 5 (iii).

(c) Suppose now that $m = n = \infty$. Let $B \in E_A$. We shall find subspaces $B_\epsilon \in D_A$ ($\varepsilon > 0$) such that $\lim_{\epsilon \to 0} \delta(B, B_\epsilon) = 0$.

Consider first the case that $A \cap B = \{0\}$, $\overline{A + B} = H$. Then $A^\perp \cap B^\perp = \{0\}$. In view of Lemma 6 there exists a selfadjoint operator $Z \in L(H)$ such that $\operatorname{Ker} Z|_{A^\perp} = \{0\}$ and $Z(A^\perp) = B$. Write

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix} : A \oplus A^\perp \to A \oplus A^\perp.$$

The operator

$$\tilde{Z} = \begin{bmatrix} Z_2 \\ Z_3 \end{bmatrix} : A^\perp \to H$$

is left invertible. Therefore, there exists $\varepsilon_0 > 0$ such that every operator

$$\tilde{X} = \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} : A^\perp \to H$$

satisfying $\|X_i - Z_i\| < \varepsilon_0$ ($i = 2, 3$) is also left invertible (see [6, Proposition

9°, p. 14]). Moreover, from the same [6, Proposition 9°] and inequality (6) it follows that

$$\operatorname{Im} \tilde{X} \to \operatorname{Im} \tilde{Z} = B \tag{9}$$

as $\max_{i=2,3} \|X_i - Z_i\| \to 0$.

On the other hand, $\operatorname{Im} \tilde{X}$ is a direct complement to A if and only if X_3 is invertible. Since $Z_3 = Z_3^*$, for every $\varepsilon > 0$ there exists an invertible operator $X_{3,\epsilon} \in L(A^\perp)$ such that $\|Z_3 - X_{3,\epsilon}\| < \varepsilon$ (this can be easily shown using the spectral decomposition of Z_3). Now put

$$B_\epsilon = \operatorname{Im} \begin{bmatrix} Z_2 \\ X_{3,\epsilon} \end{bmatrix};$$

then $B_\epsilon \in D_A$ ($\varepsilon > 0$) and, by (9), $\lim_{\epsilon \to 0} \delta(B, B_\epsilon) = 0$.

Suppose now $A \cap B \neq \{0\}$. It will suffice to find a subspace $\tilde{B}_\epsilon$ for every $\varepsilon > 0$ such that $\lim_{\epsilon \to 0} \delta(B, \tilde{B}_\epsilon) = 0$ and $A \cap \tilde{B}_\epsilon = \{0\}$, $\overline{A + \tilde{B}_\epsilon} = H$. Since $B \in E_A$, the subspaces $A \cap B$ and $H \ominus \overline{(A+B)}$ are isomorphic. Assume for definiteness that $\dim A \cap B = \dim H \ominus \overline{(A+B)} = \infty$. Let $e_1, e_2, \ldots,$ (resp. $f_1, f_2, \ldots,$) be an orthonormal basis in $A \cap B$ (resp. $H \ominus \overline{(A+B)}$), and put

$$\tilde{B}_\epsilon = (B \ominus (A \cap B)) \dot{+} M_\epsilon, \tag{10}$$

where $M_\epsilon = \operatorname{span}\{e_i + \varepsilon f_i\}_{i=1}^\infty$, $\varepsilon > 0$. Here and in the sequel $\operatorname{span}\{g_i\}_{i=1}^\infty$ denotes the closure of the linear set spanned by the vectors $g_1, g_2, \ldots$. Let us check first that the sum in (10) is indeed direct (for $\varepsilon > 0$ small enough). By Proposition 5 (i) it is sufficient to show that

$$\delta(A \cap B, M_\epsilon) \leqslant \varepsilon. \tag{11}$$

Observe that $\{e_i + \varepsilon f_i\}_{i=1}^\infty$ is an orthogonal basis in M_ϵ and $\|e_i + \varepsilon f_i\|^2 = 1 + \varepsilon^2$. Pick $x = \sum_{i=1}^\infty \alpha_i e_i \in A \cap B$, such that $\|x\|^2 = \sum_{i=1}^\infty |\alpha_i|^2 = 1$. Then

$$\operatorname{dist}(x, M_\epsilon) \leqslant \left\| x - \sum_{i=1}^\infty \alpha_i (e_i + \varepsilon f_i) \right\| = \varepsilon \left\| \sum_{i=1}^\infty \alpha_i f_i \right\| = \varepsilon.$$

On the other hand, pick $y = \sum_{i=1}^\infty \beta_i (e_i + \varepsilon f_i) \in M_\epsilon$ such that $\|y\|^2 = (1+\varepsilon^2) \sum_{i=1}^\infty |\beta_i|^2 = 1$. Then

$$\operatorname{dist}(y, A \cap B) \leqslant \left\| y - \sum_{i=1}^\infty \beta_i e_i \right\| = \varepsilon \left(\sum_{i=1}^\infty |\beta_i|^2 \right)^{1/2}$$
$$= \varepsilon (1+\varepsilon^2)^{-1/2} < \varepsilon.$$

So (11) follows, and (10) is indeed a direct sum. Inequality (11) together with Proposition 5 (ii) shows also that $\lim_{\epsilon \to 0} \delta(B, \tilde{B}_\epsilon) = 0$.

It remains to check that $A \cap \tilde{B}_\epsilon = \{0\}$, $\overline{A + \tilde{B}_\epsilon} = H$. Let $x \in A \cap \tilde{B}_\epsilon$; then

$x = b + \sum_{i=1}^{\infty} \alpha_i(e_i + \varepsilon f_i)$ for some $\alpha_i \in \mathbb{C}$, where $b \in B \ominus (A \cap B)$. Taking the orthogonal projector on $H \ominus (\overline{A + B})$, we conclude that $\sum_{i=1}^{\infty} \alpha_i \varepsilon f_i = 0$ and therefore $\alpha_1 = \alpha_2 = \cdots = 0$. Now $x = b$, which means that $x \in A \cap (B \ominus (A \cap B)) = \{0\}$. So $A \cap \tilde{B}_\varepsilon = \{0\}$. Further, clearly $A + \tilde{B}_\varepsilon \supset A + (B \ominus (A \cap B)) = A + B$, and for every i, $f_i = \varepsilon^{-1}(e_i + \varepsilon f_i) - \varepsilon^{-1} e_i \in A + \tilde{B}_\varepsilon$, so equality $\overline{A + \tilde{B}_\varepsilon} = H$ follows.

Theorem 1 is proved.

The following example shows that for $n = m = \infty$, one cannot replace E_A by $\Omega(\infty, \infty)$ in Theorem 1.

EXAMPLE 1. Let $e_1, e_2, \ldots,$ be an orthonormal basis in H, and put $A = \operatorname{span}\{e_{3i-2}\}_{i=1}^{\infty}$, $B = \operatorname{span}\{e_{3i-1}\}_{i=1}^{\infty}$. Since $A \perp B$, Proposition 5 (ii) implies that the inequality $\delta(A \dotplus B_\varepsilon,\ A \oplus B) < 2\varepsilon$ holds provided $\delta(B, B_\varepsilon) < \varepsilon < \frac{1}{4}$. Since $A \oplus B \in \Omega(\infty, \infty)$, also $A \dotplus B_\varepsilon \in \Omega(\infty, \infty)$ for $\varepsilon > 0$ small enough. So $D_A \cap U = \varnothing$ for some neighborhood U of B.

5. PROOF OF THEOREM 4

Without loss of generality, we can suppose that $A \cap B = \{0\}$. $\overline{A + B} = H$. Indeed, if $A \cap B \neq \{0\}$, consider the subspaces $\tilde{A} = A \ominus (A \cap B)$ and $\tilde{B} = B \ominus (A \cap B)$ in the Hilbert space $\tilde{H} = H \ominus (A \cap B)$. Suppose we have found $\tilde{B}_\varepsilon, \tilde{C}_\varepsilon \subset \tilde{H}$ such that $\tilde{A} \dotplus \tilde{C}_\varepsilon = \tilde{B}_\varepsilon \dotplus \tilde{C}_\varepsilon = \tilde{H}$ and $\delta(\tilde{B}, \tilde{B}_\varepsilon) < \varepsilon/2$. Then put $B_\varepsilon = \tilde{B}_\varepsilon \oplus (A \cap B)$, $C_\varepsilon = \tilde{C}_\varepsilon$ to meet the requirements of Theorem 4 (cf. Proposition 5(ii)). If $A \cap B = \{0\}$, but $\overline{A + B} \neq H$, consider A and B as subspaces in the Hilbert space $H' = \overline{A + B}$. Now, if $B'_\varepsilon, C'_\varepsilon \subset H'$ are such that $A \dotplus C'_\varepsilon = B'_\varepsilon \dotplus C'_\varepsilon = H'$ and $\delta(B', B'_\varepsilon) < \varepsilon$, put $B_\varepsilon = B'_\varepsilon$, $C_\varepsilon = C'_\varepsilon \oplus (H \ominus H')$ to prove Theorem 4.

So consider the case $A \cap B = \{0\}$, $\overline{A + B} = H$. By Theorem 1, there exists a subspace $Q_\varepsilon \subset H$ such that $A \dotplus Q_\varepsilon = H$ and $\delta(B, Q_\varepsilon) < \varepsilon$. Let $e_1, e_2, \ldots,$ (resp. $f_1, f_2, \ldots,$) be the orthonormal bases in Q_ε (resp. $Q_\varepsilon^\perp$). Consider the subspace

$$Q_\varepsilon(\alpha) = \operatorname{span}\{e_1 + \alpha f_1,\ e_2 + \alpha f_2, \ldots,\}, \qquad \alpha > 0.$$

Then (cf. the proof of (11)), $\delta(Q_\varepsilon, Q_\varepsilon(\alpha)) \leqslant \alpha$.

Clearly, Q_ε is a direct complement to $Q_\varepsilon(\alpha)$.

Then the assertions of Theorem 4 hold for $C_\varepsilon = Q_\varepsilon$ and $B_\varepsilon = Q_\varepsilon(\varepsilon - \delta(B, Q_\varepsilon))$.

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